MTH 265, Final Exam, Summer 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the volume of the region below $f(x,y) = \frac{1}{x^2-y^2}$ for the region bounded by $x+y=1$, $x+y=2,\ x-y=1, x-y=4$. (18 points)

$\to \frac{1}{(x-y)(x+y)}$

$x+y=u \quad [1,2]$

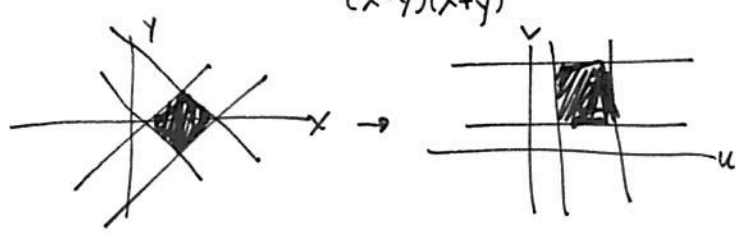
$x-y=v \quad [1,4]$

$2x = u+v$

$x = \frac{1}{2}(u+v)$

$2y = u-v$

$y = \frac{1}{2}(u-v)$



$\int_1^4 \int_1^2 \frac{1}{u}\cdot\frac{1}{v} \left|\frac{1}{2}\right| du\, dv =$

$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$

$\frac{1}{2}\int_1^4 \ln u \Big|_1^2 \frac{dv}{v} = \frac{1}{2}\ln 2 \int_1^4 \frac{1}{v}dv = \frac{1}{2}\ln 2 \cdot \ln 4$

2. Write the integrals needed to find the center of mass for the region bounded by $0 \le z \le \frac{1}{1+x^2+y^2}, x^2+y^2 \le 4, x \ge 0, y \ge 0, \rho = k\cos\theta$. Find the center of mass. (20 points)

leaving out k

$M = \int_0^{\pi/2}\int_0^2\int^{1/(1+r^2)} \cos\theta\, r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 \cos\theta \frac{r}{1+r^2}\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{2}\ln|1+r^2|\Big|_0^2 \cos\theta\, d\theta =$

$\int_0^{\pi/2} [\frac{1}{2}\ln 5 \cos\theta\, d\theta = \frac{1}{2}\ln 5 \sin\theta\Big|_0^{\pi/2} = \frac{1}{2}\ln 5$

$M_{xy} = \int_0^{\pi/2}\int_0^2\int_0^{1/(1+r^2)} \cos\theta\, z r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 \frac{1}{2}z^2\Big|_0^{1/(1+r^2)} r\cos\theta\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 \frac{1}{2}\frac{r}{(1+r^2)^2}\cos\theta\, dr\, d\theta =$

$\int_0^{\pi/2} \frac{-1}{4(r^2+1)}\Big|_0^2 \cos\theta\, d\theta = \int_0^{\pi/2}\frac{1}{5}\cos\theta\, d\theta = \frac{1}{5}\sin\theta\Big|_0^{\pi/2} = \frac{1}{5}$ $\quad \bar{z} = \frac{1/5}{\frac{1}{2}\ln 5} = \boxed{\frac{2}{5\ln 5}}$

$M_{xz} = \int_0^{\pi/2}\int_0^2\int_0^{1/(1+r^2)} r\sin\theta\cos\theta\, r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2\int_0^{\frac{1}{1+r^2}} r^2\sin\theta\cos\theta\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 \frac{r^2}{1+r^2}\sin\theta\cos\theta\, dr\, d\theta$

$= \int_0^{\pi/2} \sin\theta\cos\theta\, (r - \arctan r\Big|_0^2 d\theta = (2-\arctan 2)\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{1}{2}(2-\arctan 2)\sin^2\theta\Big|_0^{\pi/2}$

$M_{yz} = \int_0^{\pi/2}\int_0^2\int_0^{1/(1+r^2)} \cos^2\theta\, r^2\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 \frac{r^2}{1+r^2}\cos^2\theta\, dr\, d\theta =$ $\quad = 1 - \frac{1}{2}\arctan 2$ $\quad \boxed{\bar{y} = \frac{2-\arctan 2}{\ln 5}}$

$\int_0^{\pi/2} \cos^2\theta\, (r - \arctan r)\Big|_0^2 d\theta = (2-\arctan 2)\int_0^{\pi/2}\cos^2\theta\, d\theta = \frac{1}{2}(2-\arctan 2)\int_0^{\pi/2} 1+\cos 2\theta\, d\theta =$

$\frac{1}{2}(2-\arctan 2)(\theta + \frac{1}{2}\sin 2\theta\Big|_0^{\pi/2} = \frac{\pi}{2}(2-\arctan 2)$ $\quad \boxed{\bar{x} = \frac{\pi(2-\arctan 2)}{\ \ }}$

3. Use the divergence theorem to evaluate $\int_S \int \vec{F}\cdot\vec{N}\,dS$ for $\vec{F}(x,y,z) = z^2\hat{\imath} + 2x^2\hat{\jmath} + 2xy\hat{k}$ for the closed surface bounded by $x^2+y^2=1, z=2$ and the coordinate planes. (20 points)

$\vec{\nabla}\cdot\vec{F} = 0$

$\int_0^{\pi/2}\int_0^1\int_0^2 0\,dV = 0$

4. Use the second partials test to find any characterize any critical points for $z = 4x + 6y - x^2 - y^2 + xy$. (12 points)

$z_x = 4 - 2x + y = 0 \quad y = 2x - 4 \quad 2x - 4 = 3 + \frac{1}{2}x$

$z_y = 6 - 2y + x = 0 \quad 3 + \frac{1}{2}x = y \quad 4x - 8 = 6 + x$

critical point $\left(\frac{14}{3}, \frac{16}{3}\right) \quad \frac{3x}{3} = \frac{14}{3} \quad x = \frac{14}{3}$

$y = 3 + \frac{1}{2}\left(\frac{14}{3}\right) = 3 + \frac{7}{3} = \frac{16}{3}$

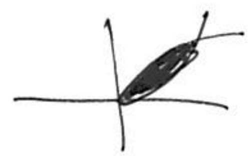
$z_{xx} = -2$

$z_{yy} = -2 \quad D = (-2)(-2) - 1^2 = 4 - 1 = 3$ max or min

$z_{xy} = 1 \quad z_{xx} = -2 \quad \cap$ concave down

critical point is a max.

5. Find the absolute extrema of the function $f(x,y) = x^2 - 3xy + 2y^2 - 6y$ over the region bounded by $y = x^2$, *and* $y = \sqrt{x}$. (18 points)



$f_x = 2x - 3y = 0 \quad x = \frac{3}{2}y$

$f_y = -3x + 4y - 6 = 0 \quad -3\left(\frac{3}{2}y\right) + 4y = 6$

$-\frac{1}{2}y = 6$

$f(x, x^2) = x^2 - 3x(x^2) + 2(x^2)^2 \quad y = 12 \to x = -18 \quad (-18, -12)$ not in region

$f'(x) = 8x^3 - 9x^2 - 10x \quad -6x^2 = x^2 - 3x^3 + 2x^4 - 6y^2 = 2x^4 - 3x^3 - 5x^2$

$= x(8x^2 - 9x - 10) \quad x = 0 \quad x = -0.68, x = 1.81$

outside region

$f(x, \sqrt{x}) = x^2 - 3x\sqrt{x} + 2(\sqrt{x})^2 - 6\sqrt{x} =$

$x^2 - 3x^{3/2} - 6x^{1/2} + 2x \Rightarrow 2x - \frac{9}{2}x^{1/2} + 2 - \frac{3}{\sqrt{x}} = f' = 0 \quad x = 4.46$

outside region

Corner points $(0,0), (1,1)$

$f(0,0) = 0$ abs. max

$f(1,1) = -6$ abs min

6. Determine the maximum height and range of a projectile fired at height of 0.5 meters above the ground, with an initial velocity of 150 meters/second at an angle of 50° with the horizontal. Use the equation $\vec{r}(t) = (v_0 \cos\theta)t\hat{\imath} + \left[h_0 + (v_0 \sin\theta)t + \frac{1}{2}gt^2\right]\hat{\jmath}$. (15 points)

150 50° 0.5 150 50 -9.8

$r(t) = 96.41814t\,\hat{\imath} + [0.5 + 114.9067t - 4.9t^2]\hat{\jmath}$

max height $y'(t) = 0$

$114.9067 - 9.8t = 0$

$t = 11.7252$ sec.

$y(11.7252) = 674.1505$ m.

range

$y(t) = 0$

$t = 23.454697$

$x(23.454697) = 2261.458$ m

7. Find the curvature of the curve $\vec{r}(t) = t^2\hat{\imath} + \cos t\,\hat{\jmath} + \sin t\,\hat{k}$ at the point $t = \frac{\pi}{3}$. What is the radius of curvature at the same point? (14 points)

$r'(t) = 2t\hat{\imath} - \sin t\,\hat{\jmath} + \cos t\,\hat{k}$ $\quad \|r'(t)\| = \sqrt{4t^2 + \cos^2 t + \sin^2 t} = \sqrt{4t^2+1}$

$r''(t) = 2\hat{\imath} - \cos t\,\hat{\jmath} - \sin t\,\hat{k}$ $\quad @\ \pi/3 = \sqrt{\frac{4\pi^2}{9}+1}$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ 2t & -\sin t & \cos t \\ 2 & -\cos t & -\sin t \end{vmatrix} = (\sin^2 t + \cos^2 t)\hat{\imath} - (-2t\sin t - 2\cos t)\hat{\jmath} + (-2t\cos t + 2\sin t)\hat{k}$$

$= 1\hat{\imath} + (2t\sin t + 2\cos t)\hat{\jmath} + (2\sin t - 2t\cos t)\hat{k}$

$@\ \pi/3 \quad \hat{\imath} + \left(\frac{\pi\sqrt{3}}{3} + 1\right)\hat{\jmath} + \left(\sqrt{3} - \frac{\pi}{3}\right)k$

$$K(\pi/3) = \frac{\sqrt{1 + \left(\frac{\pi\sqrt{3}}{3}+1\right)^2 + \left(\sqrt{3} - \frac{\pi}{3}\right)^2}}{\left(\frac{4\pi^2}{9}+1\right)^{3/2}} \approx 0.245$$

$R = \frac{1}{K} \approx 4.08$

8. Use Green's Theorem to evaluate $\int_C \overset{M}{(y - e^x)}dx + \overset{N}{(2x - \ln y)}dy$ on the path described by the boundary of circle $x^2 + y^2 = 9$ oriented counterclockwise. (15 points)

$\frac{\partial N}{\partial x} = 2 \quad \frac{\partial M}{\partial y} = 1$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1$

$\int_0^{2\pi}\int_0^3 1\,r\,dr\,d\theta$ = area of circle = 9π

9. Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^3}$. (10 points)

$x^4 = y^3$

$y = kx^{4/3}$

$$\lim_{x\to 0} \frac{x^2 \cdot kx^{4/3}}{x^4 + k^3x^4} = \lim_{x\to 0} \frac{x^{10/3}\cdot k}{x^4(1+k^3)} = \lim_{x\to 0} \frac{k}{x^{2/3}(1+k^3)} =$$

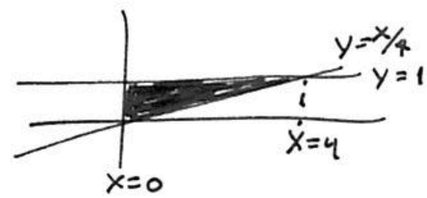
DNE

goes to $\pm\infty$ depending on k

10. Change the order of integration in $\int_0^4 \int_{x/4}^1 \frac{1}{1+y^4}\,dy\,dx$ so that it can be integrated. Then complete the integration. (12 points)

$y = \frac{x}{4} \to 4y > x$

$x > 0$



$$\int_0^1 \int_0^{4y} \frac{1}{1+y^4}\,dx\,dy =$$

$$\int_0^1 \frac{4y}{1+y^4}\,dy$$

$u = y^2$

$du = 2y\,dy$

$2\int \frac{du}{1+u^2}$

$$2\arctan y^2 \Big|_0^1$$

$$= 2\arctan 1 = 2\cdot \pi/4 = \pi/2$$

11. Evaluate the line integral $\int_C y(x+z)ds$ on the path $\vec{r}(t) = \sqrt{t}\hat{\imath} + t^2\hat{\jmath} + 2t\hat{k}$ on $[1,4]$. (12 points)

$$\int_1^4 t^2(\sqrt{t} + 2t)\sqrt{\frac{1}{4t} + 4t^2 + 4}\,dt$$

$r'(t) = \frac{1}{2\sqrt{t}}\hat{\imath} + 2t\hat{\jmath} + 2\hat{k}$

$\|r'(t)\| = \sqrt{\frac{1}{4t} + 4t^2 + 4}$

$\approx 1099.251178\ldots$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

12. Sketch the gradient field for $z = x^2 + y^2 + x^2y + 4$ and use it to characterize any critical points. (18 points)

$\nabla z = \langle 2x + 2xy,\ 2y + x^2 \rangle$

$2x(1+y) = 0$

$x = 0,\ y = -1$

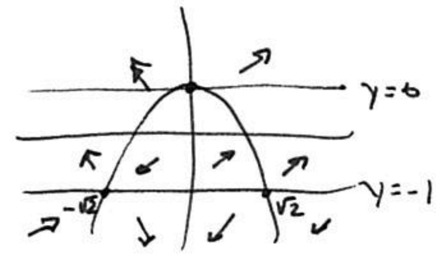
$y = -\frac{1}{2}x^2$

$-1 = -\frac{1}{2}x^2$

$2 = x^2$

$x = \pm\sqrt{2}$

(x,y)	∇z
$(1,1)$	$\langle 4,3 \rangle$
$(-1,1)$	$\langle -4,3 \rangle$
$(-1,-1/4)$	$\langle -1, 1/2 \rangle$
$(1,-1/4)$	$\langle 1, 1/2 \rangle$
$(-1/4,-1/4)$	$\langle -\frac{1}{4}, -7/16 \rangle$
$(\frac{1}{4},-1/4)$	$\langle 1/4, 9/16 \rangle$
$(-1/4,-2)$	$\langle \frac{1}{2}, -63/16 \rangle$
$(\frac{1}{4},-2)$	$\langle -\frac{1}{2}, -63/16 \rangle$
$(2,3)$	$\langle -4,-2 \rangle$
$(-2,-3)$	$\langle 4,10 \rangle$



$(0,0)$ minimum

$(\pm\sqrt{2},-1)$ saddle points

13. Write an equation of the ellipsoid $x^2 + \frac{y^2}{9} + z^2 = 1$ in parametric (surface) form. (10 points)

$r(u,v) = \cos u \sin v\,\hat{i} + 3\sin u \sin v\,\hat{j} + \cos v\,\hat{k}$

14. Find $\frac{dw}{dt}$ for $w = e^{xy} + yz, x = t^2, y = \frac{1}{t}, z = \ln t$ using the chain rule. Be sure your final answer contains only t. (12 points)

$\frac{\partial w}{\partial x} = ye^{xy} \to \frac{1}{t}e^t$ $\qquad \frac{dx}{dt} = 2t$

$\frac{\partial w}{\partial y} = xe^{xy} + z = t^2e^t + \ln t$ $\qquad \frac{dy}{dt} = \frac{-1}{t^2}$

$\frac{\partial w}{\partial z} = y = \frac{1}{t}$ $\qquad \frac{dz}{dt} = \frac{1}{t}$

$\frac{dw}{dt} = (\frac{1}{t}e^t)(2t) + (t^2e^t + \ln t)(-\frac{1}{t^2}) + (\frac{1}{t})(\frac{1}{t}) =$

$2e^t - e^t - \frac{\ln t}{t^2} + \frac{1}{t^2} = e^t + \frac{1 - \ln t}{t^2}$

15. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for $x^3y + y^2 + \sqrt{x}\sin x - 8yz = z^2$. (15 points) $F = x^3y + y^2 + \sqrt{x}\sin x - 8yz - z^2$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{3x^2y + \frac{1}{2}x^{-1/2}\sin x + \sqrt{x}\cos x}{-(-8y - 2z)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{x^3 + 2y - 8z}{-(-8y - 2z)}$$

16. Find the position vector for $\vec{a}(t) = \frac{1}{t+1}\hat{\imath} - 4\hat{k}, \vec{v}(0) = \hat{\imath} - 2\hat{\jmath} + \hat{k}, \vec{r}(0) = 3\hat{\imath}$. (15 points)

$\int \frac{1}{t+1}\, dt = \ln|t+1| + C_1 = 1 \Rightarrow C_1 = 1$

$\int 0\, dt = C_2 \Rightarrow -2 \quad C_2 = -2$

$\int -4\, dt = -4t + C_3 = 1 \quad C_3\ 1$

$v(t) = [(\ln|t+1|) + 1]\hat{\imath} - 2\hat{\jmath} + (1 - 4t)\hat{k}$

$\int \ln|t+1| + 1\, dt = (t+1)\ln(t+1) + C_1 = 3 \Rightarrow C_1 = 3$

$\int -\cancel{2}\, dt = -2t + C_2 = 0 \Rightarrow C_2 = 0$

$\int 1 - 4t\, dt = t - 2t^2 + C_3 = 0 \Rightarrow C_3 = 0$

$r(t) = [(t+1)\ln|t+1| + 2]\hat{\imath} - 3t\hat{\jmath} + (t - 2t^2)\hat{k}$

17. Find the equation of the tangent plane to the surface $x^2z^2 = y^2$ at $(1,4,-2)$. (12 points)

$F = x^2z^2 - y^2 = 0$

$\nabla F = \langle 2xz^2, -2y, 2x^2z \rangle$

$\nabla F(1,4,-2) = \langle 8, -8, -4 \rangle$

plane:

$8(x-1) - 8(y-4) - 4(z+2) = 0$

18. Convert the triple integral $\int_{-6}^{6}\int_{0}^{\sqrt{36-x^2}}\int_{x^2+y^2}^{\sqrt{36-x^2-y^2}} \frac{\ln\sqrt{x^2+y^2+z^2+1}}{x^2+y^2+z^2}\, dz dy dx$ to spherical coordinates and then complete the integration. Describe the region being integrated (over). (15 points)

$\int_0^{\pi}\int_0^{0.4024}\int_0^{6} \frac{\ln|\rho^2+1|}{\cancel{\rho^2}} \cdot \cancel{\rho^2} \sin\varphi \, d\rho\, d\varphi\, d\theta +$

$\int_0^{\pi}\int_{0.4024}^{\pi/2}\int_0^{\cot\varphi\csc\varphi} \frac{\ln|\rho^2+1|}{\cancel{\rho^2}} \cancel{\rho^2} \sin\varphi \, d\rho\, d\varphi\, d\theta =$

$\int_0^{\pi}\int_0^{0.4024} (\ln 7 - 7)\sin\varphi \, d\varphi\, d\theta +$

$\int_0^{\pi}\int_{0.4024}^{\pi/2} \cot\varphi \, \cancel{\csc\varphi} \ln|\csc\varphi\cot\varphi| \, \cancel{\sin\varphi} \, d\rho\, d\theta =$

$\int_0^{\pi} 0.52889 \, d\theta + \int_0^{\pi} 0.5074897 \, d\theta =$

$\int_0^{\pi} 1.03638 \, d\theta \approx 3.25588\ldots$

$\cancel{\rho}\cos\varphi = \rho^{\cancel{2}}\sin^2\varphi$

$\frac{\cos\varphi}{\sin\varphi} \cdot \sin\varphi = \rho$

$\rho = \cot\varphi\csc\varphi$

$x^2+y^2 = \sqrt{36 - x^2 - y^2}$

$r^2 = \sqrt{36 - r^2}$

$r^4 = 36 - r^2$

$r^4 + r^2 - 36 = 0$

$r^2 = 5.52079\ldots$

$r = 2.3496$

$r^2 = \rho^2\sin^2\varphi$

$\frac{r^2}{} = 0.15335 = \sin^2\varphi$

$\varphi = 23.054^\circ$

≈ 0.4024 radians

19. Determine if the vector field $\vec{F}(x,y,z) = (\cos y + 3x^2)\hat{\imath} - (x\sin y + z)\hat{\jmath} - y\hat{k}$ is conservative. Find the potential function. (12 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos y + 3x^2 & -x\sin y - z & -y \end{vmatrix} = (-1+1)\hat{\imath} - (0-0)\hat{\jmath} + (-\sin y + \sin y)\hat{k}$$

$$= \vec{0}$$

Conservative

$$\int \cos y + 3x^2\, dx = x\cos y + x^3 + f(y,z)$$

$$\int -x\sin y - z\, dy = x\cos y - yz + g(x,z)$$

$$\int -y\, dz = -yz + h(x,y)$$

$$\varphi = x\cos y + x^3 - yz + K$$

Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2 + y^2 = r^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

Spherical

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$\cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \phi$$
$$x^2 + y^2 = \rho^2\sin^2\phi = r^2$$

Dels

$$\frac{\partial}{\partial x} = \text{ partial derivative with respect to } x$$
$$\nabla f = grad\ f$$
$$\nabla^2 f = \nabla \cdot (\nabla f) = Laplacian\ of\ f$$
$$\nabla \cdot \vec{F} = div\ \vec{F}$$
$$\nabla \times \vec{F} = curl\ \vec{F}$$

Misc

$$ds = \|\vec{r}'(t)\|dt$$
$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{1}{R}$$
$$\vec{N} = \nabla F = \vec{r}_u \times \vec{r}_v$$
$$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$