5/30/2024

Double Integrals and Volume

Finish the fundamental theorem of line integrals: example.

$$\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy$$

On any path from point (1,0) to (2,1). (this implying that the field is conservative)

$$\vec{F}(x,y) = \langle 2xe^{-y}, 2y - x^2e^{-y}\rangle$$

$$\int 2xe^{-y}dx = x^2e^{-y} + g(y)$$

$$\int 2y - x^2e^{-y}dy = y^2 + x^2e^{-y} + h(x)$$

$$f(x,y) = x^2e^{-y} + y^2$$

$$\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy = f(2,1) - f(1,0) = 4e^{-1} + 1 - (1+0) = \frac{4}{e}$$

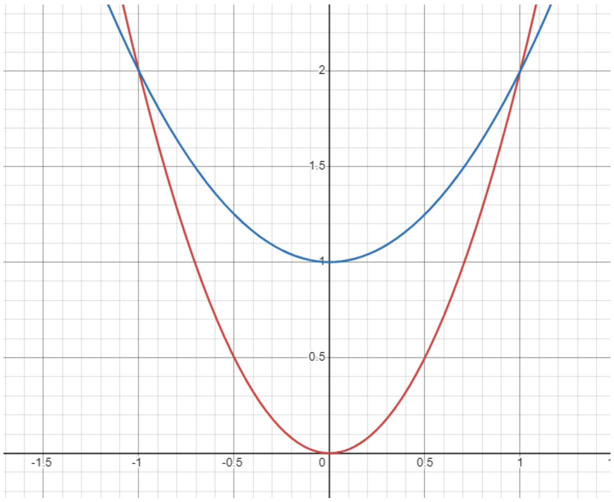
Double Integrals
Single integrals in Calc 1 represented the area under a curve. Double integrals going to represent the volume under a surface.

Recall from deriving the one variable integral, we split up x interval into several smaller segments and constructed an estimate with rectangles. The width of the smaller segments was the width of the rectangle, and the height of the rectangle was the height of the function (right-hand end, the left-hand end, or the midpoint, etc.) The idea was that as we took more rectangles, the estimate would get better until, in the limit, the answer was exact.

In the volume case, we have a surface (z=f(x,y)) that represents the height of the function hovering over some region in the xy-plane. We are dividing up both x and y into smaller segments (m for x direction and n for the y direction), what we get is little rectangles in the plane, the volume underneath the surface is given by the width (in x) and the depth (in y) and the height (in z) which is estimated from the function (at some point inside the rectangle on the plane). The estimate of the volume is based on a rectangular box, volume for each segment is $f(x_i, y_j)\Delta x\Delta y = hwl$.
Then add up the rectangular boxes to get the estimate for the total volume.

The function we are integrating represents the surface in the z-direction, as a function of x and y.
The limits of integration in x and y are the bounds on the region in the plane.

Example.
Find the area in the plane bounded by the functions $y = 2x^2, y = 1 + x^2$
Red curve is $y = 2x^2$ and blue is $y = 1 + x^2$. Intersections are at -1, 1



$$\int_a^b \int_{g(x)}^{f(x)} f(x,y)\, dA = \int_a^b \int_{g(x)}^{f(x)} f(x,y)\, dydx = \int_c^d \int_{s(y)}^{r(y)} f(x,y)\, dxdy$$

$$\int_a^b \int_{g(x)}^{f(x)} f(x,y)\, dydx = \int_a^b \left[\int_{g(x)}^{f(x)} f(x,y)dy\right] dx$$

a, b are the limits in x, and $f(x), g(x)$ are the bounds on the functions for the region, the top and bottom functions.

If we want to use a double integral to calculate just the area, we set f(x,y)=1.

Area: $\int_a^b \int_{g(x)}^{f(x)} 1\, dydx$

Area:

$$\int_{-1}^{1} \int_{2x^2}^{1+x^2} 1\, dy\, dx = \int_{-1}^{1} [y]_{2x^2}^{1+x^2} dx = \int_{-1}^{1} 1 + x^2 - (2x^2)dx = \int_{-1}^{1} 1 - x^2\, dx = 2\int_0^1 1 - x^2 dx =$$

$$2(x - \frac{x^3}{3}\Big|_0^1 = 2\left[1 - \frac{1}{3}\right) = 2\left(\frac{2}{3}\right) = \frac{4}{3}$$

$$\int_{-1}^{1} f(x) - g(x)dx = \frac{4}{3}$$

Volume will come when the function is not 1, but some $f(x,y)$

Suppose we want to find the volume of the region under the surface $f(x,y) = x + 2y$, above the region in the plane bounded by $y = 2x^2, y = 1 + x^2$.

$$Volume = \int_{-1}^{1}\int_{2x^2}^{1+x^2} x + 2y\, dydx$$

$$\int_{-1}^{1}\left[\int_{2x^2}^{1+x^2}(x+2y)dy\right]dx = \int_{-1}^{1}[xy+y^2]_{2x^2}^{1+x^2}dx = \int_{-1}^{1}x(1+x^2)+(1+x^2)^2 - x(2x^2)-(2x^2)^2dx$$

$$= \int_{-1}^{1}x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4dx = \int_{-1}^{1}1 + x + 2x^2 - x^3 - 3x^4\, dx =$$

$$\int_{-1}^{1}1+2x^2-3x^4dx + \int_{-1}^{1}x - x^3dx = 2\int_{0}^{1}1+2x^2-3x^4dx = 2\left[x+\frac{2x^3}{3}-\frac{3x^5}{5}\right]_0^1 = 2\left[1+\frac{2}{3}-\frac{3}{5}\right]$$

$$\frac{32}{15}$$

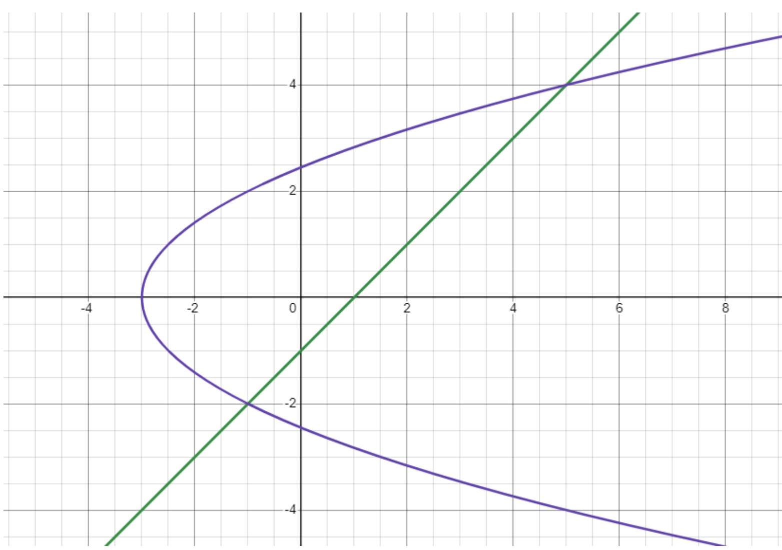
To find the average height of the surface on this region, divide the total volume by the area in the plane.

$$\bar{f} = \frac{\frac{32}{15}}{\frac{4}{3}} = \frac{32}{15}\times\frac{3}{4} = \frac{8}{5}$$

$$\bar{f} = \frac{V}{A} = \frac{1}{A}\int_a^b\int_{g(x)}^{f(x)} f(x,y)dA$$

Typically way of setting up these double integrals integrate in y first (using the top and bottom functions as the limits), and then in x last. If you integrate in x first, the inside functions are the right-hand and left-hand boundaries (x=f(y)) and the outer limits are constants in y. This is equivalent to integrating in one variable with horizontal rectangles instead of vertical ones.

Find the volume under the surface $z = xy$ above the region in the plane bounded by $y = x - 1$, $y^2 = 2x + 6$



y=x-1 becomes x=y+1

Solve for x in terms of y:

$$y^2 = 2x + 6 \to y^2 - 6 = 2x \to \frac{y^2}{2} - 3 = x$$

$$\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dxdy = \int_{-2}^{4}\left[\int_{\frac{y^2}{2}-3}^{y+1} xy\,dx\right]dy = \int_{-2}^{4}\left[\frac{x^2y}{2}\right]_{\frac{y^2}{2}-3}^{y+1} dy = \frac{1}{2}\int_{-2}^{4}(y+1)^2y - \left(\frac{y^2}{2}-3\right)^2 y\,dy =$$

$$\frac{1}{2}\int_{-2}^{4}(y^2+2y+1)y - \left(\frac{y^4}{4} - 3y^2 + 9\right)y\,dy = \frac{1}{2}\int_{-2}^{4} y^3 + 2y^2 + y - \frac{y^5}{4} + 3y^3 - 9y\,dy =$$

$$\frac{1}{2}\int_{-2}^{4} -\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\,dy = \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + \frac{2}{3}y^3 - 4y^2\right]_{-2}^{4} =$$

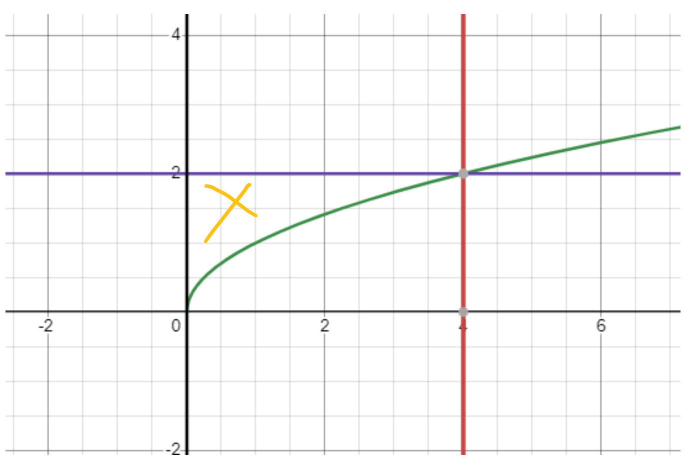
$$\frac{1}{2}\left[-\frac{512}{3} + 256 + \frac{128}{3} - 64 - \left(-\frac{8}{3} + 16 - \frac{16}{3} - 16\right)\right] = 36$$

Switching the order the of integration in a double integral.

$$\int_{0}^{4}\int_{\sqrt{x}}^{2}\frac{1}{y^3+1}\,dydx$$

The limits are the boundary in the plane of the region we are integrating over. We need to graph that region in order to change the order of integration. The two inside limits are $y = \sqrt{x}, y = 2$, and the outside limits are $x = 0, x = 4$.

Often, one of the outside limits is “extra” in the sense that it is derived from the intersection of the other two curves.



$$y = \sqrt{x} \to y^2 = x$$

$$\int_{0}^{2}\int_{0}^{y^2}\frac{1}{y^3+1}\,dxdy = \int_{0}^{2}\left[\frac{1}{y^3+1}(x)\right]_{0}^{y^2} dy = \int_{0}^{2}\frac{y^2}{y^3+1}\,dy = \frac{1}{3}\ln(y^3+1)\Big|_{0}^{2} = \frac{1}{3}[\ln 9 - \ln 1]$$

$$= \frac{\ln 9}{3}$$

$$u = y^3 + 1, du = 3y^2 dy \rightarrow \frac{1}{3} du = y^2 dy$$

$$\int \frac{1}{u}\left(\frac{1}{3} du\right) = \frac{1}{3}\ln(u)$$

Exam #1 is Monday.