6/6/2024

Tangent Vectors, Normal Vectors, Bi-Normal Vectors/Normal Planes (13.3)
Directional Derivatives (14.6)

Dealing with space curves : $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

The tangent is related to the derivative.
A tangent vector to a space curve is related the derivative $\vec{r}'(t)$
Unit tangent vector is just the direction that the tangent line is pointed in, while $\vec{r}'(t)$ you can think of as the velocity vector that comes with magnitude as well as direction.

The unit tangent will be just one vector that represents a coordinate system that moves along the curve and defines the orientation of an object on that path. The full 3D coordinates will depend on the tangent, and the normal, and the bi-normal vector.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Example.
Find the unit tangent vector to the curve $\vec{r}(t) = \langle 12t, 8t^{\frac{3}{2}}, 3t^2\rangle$

$$\vec{r}'(t) = \langle 12, 8\left(\frac{3}{2}\right)t^{\frac{1}{2}}, 6t\rangle = \langle 12, 12t^{\frac{1}{2}}, 6t\rangle$$

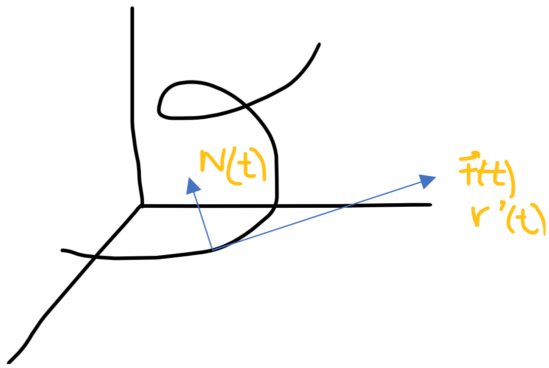
$$\|\vec{r}'(t)\| = \sqrt{144 + 144t + 36t^2} = 6\sqrt{4 + 4t + t^2} = 6\sqrt{(2+t)^2} = 6(2+t)$$

(can be thought of as the speed)

$$\vec{T}(t) = \frac{\langle 12, 12t^{\frac{1}{2}}, 6t\rangle}{6(2+t)} = \langle \frac{2}{(t+2)}, \frac{2\sqrt{t}}{t+2}, \frac{t}{t+2}\rangle$$

Unit tangent vector points in the direction of the tangent, but has magnitude = 1.

Normal Vector (unit normal)
(perpendicular—perpendicular to the curve and to the tangent vector)
The principal unit normal vector points inward to the inside of any curve in the space curve.



$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Example.
Find the unit tangent and unit normal vectors for $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 1\rangle$$

$$\|\vec{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{4+1} = \sqrt{5}$$

$$\vec{T}(t) = \frac{\langle -2\sin t, 2\cos t, 1\rangle}{\sqrt{5}}$$

To find the unit normal vector, take the derivative of the unit tangent vector.

$$\vec{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\cos t, -2\sin t, 0\rangle$$

This is not a unit vector.

$$\|k\vec{v}\| = |k|\|\vec{v}\|$$

$$\|\langle -2\cos t, -2\sin t, 0\rangle\| = \sqrt{4\cos^2 t + 4\sin^2 t} = \sqrt{4} = 2$$

$$\|\vec{T}'(t)\| = \frac{2}{\sqrt{5}}$$

$$\vec{N}(t) = \frac{\frac{1}{\sqrt{5}}\langle -2\cos t, -2\sin t, 0\rangle}{\frac{2}{\sqrt{5}}} = \frac{1}{\sqrt{5}}\langle -2\cos t, -2\sin t, 0\rangle\left(\frac{\sqrt{5}}{2}\right) = \langle -\cos t, -\sin t, 0\rangle$$

Bi-normal vector (unit vector)

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\frac{2}{\sqrt{5}}\sin t & \frac{2}{\sqrt{5}}\cos t & \frac{1}{\sqrt{5}} \\ -\cos t & -\sin t & 0 \end{vmatrix} =$$

$$\langle \left(0 + \frac{1}{\sqrt{5}}\sin t\right), -\left(0 + \frac{1}{\sqrt{5}}\cos t\right), \frac{2}{\sqrt{5}}\sin^2 t + \frac{2}{\sqrt{5}}\cos^2 t \rangle = \langle \frac{1}{\sqrt{5}}\sin t, -\frac{1}{\sqrt{5}}\cos t, \frac{2}{\sqrt{5}} \rangle$$

Unit vector.

Normal plane:
A plane that is perpendicular to the direction of motion on a space curve.
Contains both the normal and the binormal vectors.
The tangent vector is perpendicular to the normal plane.

To create the plane, you can use either $\vec{T}(t)$ or $\vec{r}'(t)$

Choose a point to evaluate the function and find the normal plane: $t = \frac{\pi}{4}$
(sometimes these problems will give you a point in (x,y,z), and you have to figure out what the t value is).

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$$

Evaluate the t-value in the curve to find a point on the normal plane.

$$\vec{r}\left(\frac{\pi}{4}\right) = \langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \rangle$$

$$\vec{T}(t) = \frac{\langle -2\sin t, 2\cos t, 1 \rangle}{\sqrt{5}}$$

$\vec{T}\left(\frac{\pi}{4}\right) = \langle -\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, \frac{1}{\sqrt{5}} \rangle$ or $\vec{r}'\left(\frac{\pi}{4}\right) = \langle -\sqrt{2}, \sqrt{2}, 1 \rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Tangent vector gives the a,b,c and the original curve gives us the point.

$$-\sqrt{2}(x - \sqrt{2}) + \sqrt{2}(y - \sqrt{2}) + 1\left(z - \frac{\pi}{4}\right) = 0$$

Example in 2D:

$$\vec{r}(t) = \langle t, \frac{1}{2}t^2 \rangle$$

Unit tangent vector:

$$\vec{r}'(t) = \langle 1, t \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{1 + t^2}$$

$$\vec{T}(t) = \frac{\langle 1, t \rangle}{\sqrt{1 + t^2}} = (1 + t^2)^{-\frac{1}{2}} \langle 1, t \rangle$$

$$\vec{T}'(t) = \left(-\frac{1}{2}\right)(1+t^2)^{-\frac{3}{2}}(2t)\langle 1,t\rangle + (1+t^2)^{-\frac{1}{2}}\langle 0,1\rangle =$$

$$\frac{-t\langle 1,t\rangle}{(1+t^2)^{\frac{3}{2}}} + \frac{\langle 0,1\rangle}{(1+t^2)^{\frac{1}{2}}} = \frac{\langle -t,-t^2\rangle}{(1+t^2)^{\frac{3}{2}}} + \frac{(1+t^2)\langle 0,1\rangle}{(1+t^2)^{\frac{3}{2}}} = \frac{\langle -t,-t^2\rangle}{(1+t^2)^{\frac{3}{2}}} + \frac{\langle 0,1+t^2\rangle}{(1+t^2)^{\frac{3}{2}}} = \frac{\langle -t,1\rangle}{(1+t^2)^{\frac{3}{2}}}$$

$$\left\|\vec{T}'(t)\right\| = (1+t^2)^{-\frac{3}{2}}\|\langle -t,1\rangle\| = (1+t^2)^{-\frac{3}{2}}\sqrt{t^2+1}$$

$$\vec{N}(t) = \frac{(1+t^2)^{-\frac{3}{2}}\langle -t,1\rangle}{(1+t^2)^{-\frac{3}{2}}\sqrt{t^2+1}} = \frac{\langle -t,1\rangle}{\sqrt{t^2+1}}$$

Compare the unit normal vector with the unit tangent vector:

$$\vec{T}(t) = \frac{\langle 1,t\rangle}{\sqrt{1+t^2}} \ \ vs. \vec{N}(t) = \frac{\langle -t,1\rangle}{\sqrt{t^2+1}}$$

In 2D, if the unit tangent vector has the form $\langle x(t), y(t)\rangle$, the principal unit normal vector will be either:

$$\langle -y(t), x(t)\rangle \ or \ \langle y(t), -x(t)\rangle$$

The components will flip and one will change sign.

Binormal vector in the 2D case, only the z-coordinate survives in the cross product (all other terms are multiplied by 0). $\vec{B}(t) = \hat{k} = \langle 0,0,1\rangle$

Directional Derivative.

When we took the partial derivatives, we were interested only in the part of the derivative that was changing in the direction of x only or y only (etc.)

What if we want to travel over a surface in a direction that is not just the coordinate axes? How do we know how the surface (function value) is changing in that direction?

Directional derivatives: how the function is changing when moving in a particular (unit) direction.

$$D_{\vec{u}}f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

It's important the $\vec{u}$ be a unit vector. Sometimes problems will give you problems that are not unit vectors and you will need to calculate the unit vector before continuing.

In 2D, sometimes the direction will be given as an angle, in that case use $\langle \cos t, \sin t\rangle$ as the unit vector.

You should get a number out of this. Evaluate the function at a specified point.

Example.

$$f(x,y) = x^3y^4 + x^4y^3, (1,1), \theta = \frac{\pi}{6}$$

$$\nabla f(x,y) = \langle 3x^2y^4 + 4x^3y^3, 4x^3y^3 + 3x^4y^2\rangle$$
$$\nabla f(1,1) = \langle 7,7\rangle$$

$$\vec{u} = \langle \cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right) \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

$$D_{\vec{u}} f(1,1) = \langle 7,7 \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \frac{7\sqrt{3}}{2} + \frac{7}{2} = \frac{7\sqrt{3}+7}{2} \approx 9.562 \ldots$$

Example.

$$f(x,y,z) = xe^y + ye^z + ze^x, (0,0,0), \vec{v} = \langle 5,1,-2 \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 5,1,-2 \rangle}{\sqrt{25+1+4}} = \langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}} \rangle$$

$$\nabla f(x,y,z) = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x \rangle$$
$$\nabla f(0,0,0) = \langle 1,1,1 \rangle$$

$$D_{\vec{u}} f(0,0,0) = \langle 1,1,1 \rangle \cdot \langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}} \rangle = \frac{5}{\sqrt{30}} + \frac{1}{\sqrt{30}} - \frac{2}{\sqrt{30}} = \frac{4}{\sqrt{30}}$$

The direction of maximum increase of the directional derivative is in the direction of the gradient. (direction would be specified by a unit vector in that direction).
The direction of maximum decrease of the directional derivative in the direction of the negative of the gradient.

Magnitude of the greatest increase is $\|\nabla f\|$, and the magnitude of the greatest decrease is the same.

Next class (Monday) is the second exam.
On Tuesday, we'll pick up with tangent planes of surfaces.

You may be asked at some point to find the equation of a tangent line or a normal line. Recall that the parametric version of a line is $\vec{r}(t) = \langle at + x_0, bt + y_0, ct + z_0 \rangle$ and the symmetric version of the line is $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ (but of course, you can't use this form if a,b or c are 0).