MTH 265, Quiz #19, Summer 2024 Name KEY

Instructions: Show all work. Use exact answers unless otherwise asked to round.

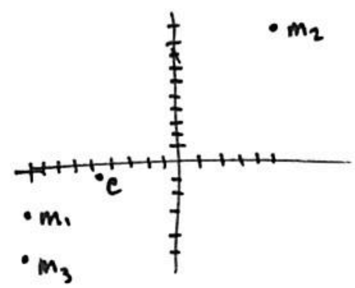
1. Three point-masses lie in a plane and are connected by masses rods so that they cannot move relative to each other. The masses and their positions are:

$$m_1 = 2.1\ kg\ at\ (-9,\ -3)$$
$$m_2 = 2.6\ kg\ at\ (5,\ 10)$$
$$m_3 = 4.1\ kg\ at\ (-9,\ -5)$$

with distances in meters. Calculate the location of this system's center of mass.

$2.1(-9) + 2.6(5) + 4.1(-9) = -42.8$ $\qquad m_1 + m_2 + m_3 = 8.8$

$\frac{-42.8}{8.8} \approx -4.86 = -\frac{107}{22}$



$2.1(-3) + 2.6(10) + 4.1(-5) = -0.8$

$\frac{-0.8}{8.8} \approx -0.0909 = -\frac{1}{11}$

$(\bar{x}, \bar{y}) = \left(-\frac{107}{22}, -\frac{1}{11}\right)$

2. Find the center of mass of the region bounded by $z = 1 - x^2 - y^2, z = 0$, with mass density $\rho = k(x^2 + y^2)$.

$z = 1 - r^2 \qquad r = 1$
$\rho = kr^2$

$M = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} kr^2 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} kr^3\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^1 z r^3\Big|_0^{1-r^2} dr\,d\theta =$

$= k\int_0^{2\pi}\int_0^1 r^3 - r^5\,dr\,d\theta = k\int_0^{2\pi} \frac{1}{4}r^4 - \frac{1}{6}r^6\Big|_0^1 d\theta = k\,\frac{1}{12}\cdot 2\pi = k\pi/6$

$M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} kr^3 z\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^1 \frac{1}{2}z^2 r^3\Big|_0^{1-r^2} dr\,d\theta = \frac{k}{2}\int_0^{2\pi}\int_0^1 r^3\underbrace{(1-r^2)^2}_{1-2r^2+r^4} dr\,d\theta$

$= \frac{k}{2}\int_0^{2\pi}\int_0^1 r^3 - 2r^5 + r^7\,dr\,d\theta = \frac{k}{2}\int_0^{2\pi} \frac{1}{4}r^3 - \frac{2}{6}r^6 + \frac{1}{8}r^8\Big|_0^1 d\theta = \frac{k}{48}\int_0^{2\pi} d\theta = \frac{k}{48}\cdot 2\pi = \frac{k\pi}{24}$

$M_{xz} = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} kr^3\cdot r\cos\theta\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^1 kr^4\cos\theta\, z\Big|_0^{1-r^2} dr\,d\theta =$

$k\int_0^{2\pi}\int_0^1 r^4 - r^6\,dr\,d\theta \cdot\cos\theta = k\int_0^{2\pi} \frac{1}{5}r^5 - \frac{1}{7}r^7\Big|_0^1 d\theta\,\cos\theta = \frac{2k}{35}\int_0^{2\pi}\cos\theta\,d\theta = \qquad 0$

$M_{yz} = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} kr^3 r\sin\theta\,dz\,dr\,d\theta = k\int_0^{2\pi}\int_0^1 r^4\sin\theta\, z\Big|_0^{1-r^2} dr\,d\theta =$

$k\int_0^{2\pi}\int_0^1 (r^4 - r^6)\sin\theta\,d\theta = k\int_0^{2\pi} \frac{1}{5}r^5 - \frac{1}{7}r^7\Big|_0^1 \sin\theta\,d\theta = \frac{2k}{35}\int_0^{2\pi}\sin\theta\,d\theta = 0$

$\bar{x} = \frac{M_{yz}}{M} = 0 \qquad \bar{y} = \frac{M_{xz}}{M} = 0 \qquad \bar{z} = \frac{M_{xy}}{M} = \frac{k\pi}{24}\cdot\frac{6}{k\pi} = \frac{1}{4}$